Distribution of parametric conductance derivatives of a quantum dot

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The conductance G of a quantum dot with single-mode ballistic point contacts depends sensitively on external parameters X, such as gate voltage and magnetic field. We calculate the joint distribution of G and dG/dX by relating it to the distribution of the Wigner-Smith time-delay matrix of a chaotic system. The distribution of dG/dX has a singularity at zero and algebraic tails. While G and dG/dX are correlated, the ratio of dG/dX and $\sqrt{G(1-G)}$ is independent of G. Coulomb interactions change the distribution of dG/dX, by inducing a transition from the grand-canonical to the canonical ensemble. All these predictions can be tested in semiconductor microstructures or microwave cavities.

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Parametric fluctuations in quantum systems with a chaotic classical dynamics are of fundamental importance for the characterization of mesoscopic systems. The fluctuating dependence of an energy level $E_j(X)$ on an external parameter X, such as the magnetic field, has received considerable attention [1]. A key role is played by the "level velocity" dE_j/dX, describing the response to a small perturbation [2–4]. In open systems, the role of the level velocity is played by the "conductance velocity" dG/dX. Remarkably little is known about its distribution.

The interest in this problem was stimulated by experiments on semiconductor microstructures known as quantum dots, in which the electron motion is ballistic and chaotic [5]. A typical quantum dot is confined by gate electrodes, and connected to two electron reservoirs by ballistic point contacts, through which only a few modes can propagate at the Fermi level. The parametric dependence of the conductance has been measured by several groups [6–8]. In the single-mode limit, parametric fluctuations are of the same order as the average, so that one needs the complete distribution of G and dG/dX to characterize the system. Knowing the average and variance is not sufficient. Analytical results are available for point contacts with a large number of modes [9–15]. In this paper, we present the complete distribution in the opposite limit of two single-mode point contacts and show that it differs strikingly from the multi-mode case considered previously.

The main differences which we have found are the following. We consider the joint distribution of the conductance G and the derivatives $\partial G/\partial V$, $\partial G/\partial X$ with respect to the gate voltage V and an external parameter X (typically the magnetic field). If the point contacts contain a large number of modes, $P(G, \partial G/\partial V, \partial G/\partial X)$ factorizes into three independent Gaussian distributions [9–12]. In the single-mode case, in contrast, we find that this distribution does not factorize and decays algebraically rather than exponentially. By integrating out G and one of the two derivatives, we obtain the conductance velocity distributions $P(\partial G/\partial V)$ and $P(\partial G/\partial X)$ plotted in Fig. 1. Both distributions have a singularity at zero velocity, and algebraic tails. A remarkable prediction of our theory is that the correlations between G, on the one hand, and $\partial G/\partial V$ and $\partial G/\partial X$, on the other hand, can be transformed away by the change of variables $G = (2e^2/h)\sin^2\theta$, where θ is the polar coordinate introduced in Ref. [13]. The derivatives $\partial\theta/\partial V$ and $\partial\theta/\partial X$ are statistically independent of θ. There exists no change of variables that transforms away the correlations between $\partial G/\partial V$ and $\partial G/\partial X$.

Another new feature of the single-mode case concerns the effect of Coulomb interactions [16,17]. In the simplest model, the strength of the Coulomb repulsion is measured by the ratio of the charging energy e^2/C (with C the capacitance of the quantum dot) and the mean level spacing Δ. In the regime $e^2/C \gg \Delta$, where most experiments are done, Coulomb interactions suppress fluctuations of the charge Q on the quantum dot as a function of V or X, at the expense of fluctuations in the electrical potential U. Since the Fermi level μ in the quantum dot is pinned by the reservoirs, the kinetic energy $E = \mu - U$ at the Fermi level fluctuates as well. Fluctuations of E can not be ignored, because the conductance is determined by E, an not by μ. An ensemble of quantum dots with fixed Q and fluctuating E behaves effectively as a canonical ensemble — rather than a grand-canonical ensemble. In the opposite regime $e^2/C \ll \Delta$, the energy E does not fluctuate on the scale of the level spacing. The ensemble is now truly grand-canonical. Fluctua-

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tions of E on the scale of Δ can be neglected in the multi-mode case, so that the distinction between canonical and grand-canonical averages is irrelevant. In the single-mode case the distinction becomes important. We will see that the distribution of the conductance velocities is different in the two ensembles. (The distribution of the conductance itself is the same.) The difference between grand-canonical and canonical averages has been studied extensively in connection with the problem of the persistent current [18–20], which is a thermodynamic property. Here we find a difference in the case of a transport property, which is more unusual [21,22].

To derive these results, we combine a scattering formalism with random-matrix theory [23]. The 2×2 scattering matrix S determines the conductance

$$G = |S_{12}|^2, \tag{1}$$

and the (unscreened) compressibilities [17]

$$\frac{\partial Q}{\partial E} = \frac{1}{2\pi i}\operatorname{tr} S^\dagger \frac{\partial S}{\partial E}, \quad \frac{\partial Q}{\partial X} = \frac{1}{2\pi i}\operatorname{tr} S^\dagger \frac{\partial S}{\partial X}. \tag{2}$$

(We measure G in units of $2e^2/h$ and Q in units of e.) Grand-canonical averages $\langle \cdots \rangle_{GC}$ and canonical averages $\langle \cdots \rangle_C$ are related by

$$\langle \cdots \rangle_C = \Delta \langle \cdots \times dQ/dE \rangle_{GC}. \tag{3}$$

The factor dQ/dE is the Jacobian to go from an average over Q in the canonical ensemble to an average over E in the grand-canonical ensemble. Conductance velocities in the two ensembles are related by

$$\left.\frac{\partial G}{\partial X}\right|_Q = \left.\frac{\partial G}{\partial X}\right|_E - \frac{\partial G}{\partial E}\frac{\partial Q}{\partial X}\left(\frac{\partial Q}{\partial E}\right)^{-1}, \tag{4}$$

where $|_Q$ and $|_E$ indicate, respectively, derivatives at constant Q (canonical) and constant E (grand-canonical). Derivatives $\partial G/\partial V$ with respect to the gate voltage are proportional to $\partial G/\partial Q$ in the canonical ensemble and to $\partial G/\partial E$ in the grand-canonical ensemble. (The proportionality coefficients contain elements of the capacitance matrix of the quantum dot plus gates.) The two derivatives are related by

$$\frac{\partial G}{\partial Q} = \frac{\partial G}{\partial E}\left(\frac{\partial Q}{\partial E}\right)^{-1}. \tag{5}$$

The problem that we face is the calculation of the joint distribution of S, $\partial S/\partial E$, and $\partial S/\partial X$. In view of the relations (3)–(5) it is sufficient to consider the grand-canonical ensemble. This problem is closely related to the old problem [24] of the distribution of the Wigner-Smith delay times $\tau_1, \ldots, \tau_N$, which are the eigenvalues of the $N \times N$ matrix $-iS^\dagger \partial S/\partial E$. (The eigenvalues are real positive numbers.) Interest in this problem has revived in connection with chaotic scattering [25–28]. The rates $\gamma_n = 1/\tau_n$ are distributed according to [28]

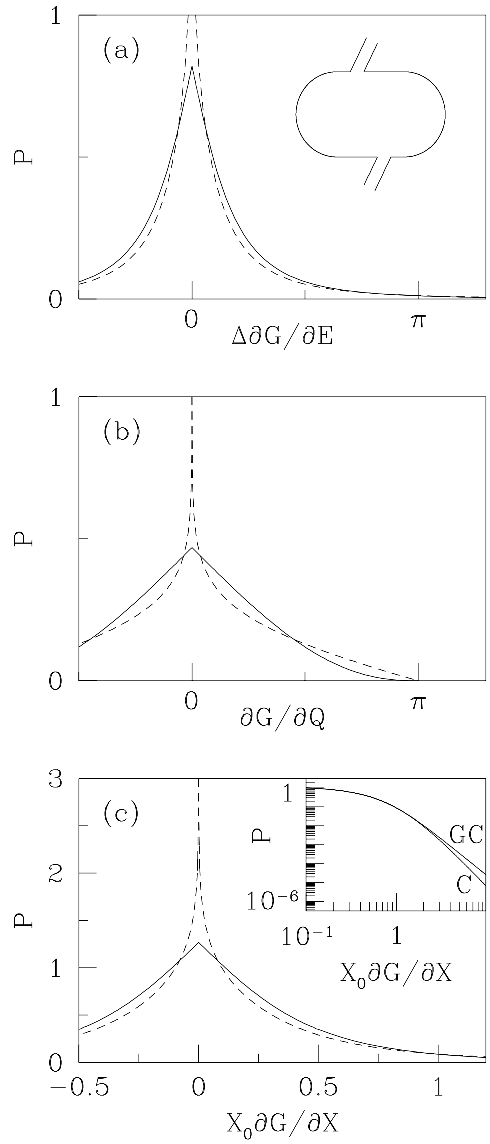

FIG. 1. Distributions of conductance velocities in a chaotic cavity with two single-mode point contacts [inset in (a)], computed from Eq. (10). Dashed curves are for $\beta = 1$ (time-reversal symmetry), solid curves for $\beta = 2$ (no time-reversal symmetry). (The case $\beta = 4$, which is similar to $\beta = 2$, is omitted for clarity.) The distribution of $\Delta\partial G/\partial E$ (grand-canonical ensemble) is shown in (a) and the distribution of $\partial G/\partial Q$ (canonical ensemble) is shown in (b). (The conductance G is measured in units of $2e^2/h$, the charge Q in units of e.) In (c) the distribution of $X_0\partial G/\partial X$ is shown for the grand-canonical ensemble (the canonical case being nearly identical on a linear scale). The inset compares the canonical (C) and grand-canonical (GC) results for $\beta = 2$ on a logarithmic scale.

$$P(\{\gamma_n\}) \propto \prod_{i<j} |\gamma_i - \gamma_j|^\beta \prod_k \gamma_k^{\beta N/2} e^{-\pi\beta\gamma_k/\Delta}. \quad (6)$$

This distribution is known in random-matrix theory as the Laguerre ensemble, because the correlation functions can be written as series over (generalized) Laguerre polynomials [29]. For $N = 1$ we recover the result of Refs. [25] and [27]. In our case $N = 2$.

To compute the conductance velocities it is not sufficient to know the delay times τ_n, but we also need to know the distribution of the eigenvectors of the time-delay matrix $-iS^\dagger \partial S/\partial E$. Furthermore, we need the distribution of $-iS^\dagger \partial S/\partial X$. The general result containing this information is [28]

$$P(S, \tau_E, \tau_X) \propto \exp\left[-\beta \,\mathrm{tr}\left(\frac{\pi}{\Delta}\tau_E^{-1} + \frac{\pi^2 X_0^2}{4\Delta^2}(\tau_E^{-1}\tau_X)^2\right)\right] \times (\det \tau_E)^{-2\beta N+3(\beta-2)/2}, \quad (7)$$

$$\tau_E = -iS^{-1/2}\frac{\partial S}{\partial E}S^{-1/2}, \ \tau_X = -iS^{-1/2}\frac{\partial S}{\partial X}S^{-1/2}. \quad (8)$$

The matrix τ_E has the same eigenvalues as the time-delay matrix, but it is more convenient because it is uncorrelated with S, while the time-delay matrix is not. By integrating out τ_E and τ_X from Eq. (7), we obtain a uniform distribution for S, as expected for a chaotic cavity [30]. The resulting distribution of the conductance [31], $P(G) \propto G^{-1+\beta/2}$, is the same in the canonical and grand-canonical ensembles, because S and dQ/dE are uncorrelated [cf. Eq. (3)]. By integrating out S, τ_X, and the eigenvectors of τ_E, we obtain the distribution (6) of the delay times. The distribution of τ_X at fixed τ_E is a Gaussian. The scale of this Gaussian is set by the parameter X_0, which has no universal value [32].

We are now ready to compute the distribution of the conductance velocities. Derivatives with respect to E and Q are related to the delay times by

$$\frac{\partial G}{\partial E} = c(\tau_1 - \tau_2)\sqrt{G(1-G)}, \quad (9a)$$

$$\frac{\partial G}{\partial Q} = 2\pi c\frac{\tau_1 - \tau_2}{\tau_1 + \tau_2}\sqrt{G(1-G)}, \quad (9b)$$

where $c \in [-1, 1]$ is a number that depends on the phases of the matrix elements of S and on the eigenvectors of τ_E. Its distribution $P(c) \propto (1-c^2)^{-1+\beta/2}$ is independent of τ_1, τ_2, and G. The derivative $\partial G/\partial X$ has a Gaussian distribution at a given value of S and τ_E, with zero mean and with variance

$$\left\langle \left(\frac{\partial G}{\partial X}\bigg|_E \right)^2 \right\rangle = \alpha \left[G(1-G)\tau_1\tau_2 + \frac{1}{2}\left(\frac{\partial G}{\partial E}\right)^2 \right],$$

$$\left\langle \left(\frac{\partial G}{\partial X}\bigg|_Q \right)^2 \right\rangle = \alpha\tau_1\tau_2 \left[G(1-G) - \frac{1}{4\pi^2}\left(\frac{\partial G}{\partial Q}\right)^2 \right],$$

where we have abbreviated $\alpha = 4\Delta^2/\pi^2 X_0^2 \beta$. Because the variance of $\partial G/\partial X$ depends on $\partial G/\partial E$ or $\partial G/\partial Q$, these conductance velocities are correlated.

From the distribution (6) of τ_1, τ_2, and the independent distributions of G and c, we calculate the joint distribution of G and its (dimensionless) derivatives $G_X = X_0 \partial G/\partial X$, $G_E = (\Delta/2\pi)\partial G/\partial E$, and $G_Q = (1/2\pi)\partial G/\partial Q$. The result in the grand-canonical and canonical ensembles is

$$P_{GC}(G, G_E, G_X) = \frac{1}{Z}\int_0^\infty dx \int_{\frac{G_E^2}{G(1-G)}}^\infty dy \frac{\left[yG - G_E^2/(1-G)\right]^{-1+\beta/2} x^{-2-2\beta}}{\sqrt{\pi(x+y)G(1-G)f(x)}} \exp\left[-\frac{2\beta}{x}\sqrt{x+y} - \frac{G_X^2}{f(x)}\right], \quad (10a)$$

$$P_C(G, G_Q, G_X) = \frac{2}{Z}\int_0^\infty dx \int_{\frac{G_Q^2}{G(1-G)}}^1 dy \frac{\left[yG - G_Q^2/(1-G)\right]^{-1+\beta/2} x^{3\beta}}{(1-y)^{(\beta+3)/2}\sqrt{\pi G(1-G)g(x)}} \exp\left[-\frac{2\beta x}{\sqrt{1-y}} - \frac{G_X^2}{g(x)}\right], \quad (10b)$$

$$f(x) = 8\beta^{-1}[xG(1-G) + 2G_E^2], \quad g(x) = 8(x^2\beta)^{-1}[G(1-G) - G_Q^2], \quad Z = 3\beta^{-3\beta-1}\Gamma(\beta/2)\Gamma(\beta)\Gamma(3\beta/2). \quad (10c)$$

By integrating out G and one of the two derivatives from Eq. (10), we obtain the conductance velocity distributions of Fig. 1. (The case $\beta = 4$ is close to $\beta = 2$ and is omitted from the plot for clarity.) The distributions have a singularity at zero derivative: A cusp for $\beta = 2$ and 4, and a logarithmic divergence for $\beta = 1$. The tails of the distributions of $\partial G/\partial X$ are algebraic in both ensembles, but with a different exponent,

$$P_{GC}(\partial G/\partial X) \propto (\partial G/\partial X)^{-\beta-2}, \quad (11a)$$

$$P_C(\partial G/\partial X) \propto (\partial G/\partial X)^{-2\beta-1}. \quad (11b)$$

The distribution of $\partial G/\partial E$ (grand-canonical ensemble) also has an algebraic tail [$\propto (\partial G/\partial E)^{-\beta-2}$], while the distribution of $\partial G/\partial Q$ (canonical ensemble) is identically zero for $|\partial G/\partial Q| \geq \pi$. In both ensembles, the second moment of the conductance velocities is finite for $\beta = 2$ and 4, but infinite for $\beta = 1$.

In conclusion, we have calculated the joint distribution of the conductance G and its parametric derivatives for a chaotic cavity, coupled to electron reservoirs by two single-mode ballistic point contacts. The distribution is fundamentally different from the multi-mode case, being highly non-Gaussian and with correlated derivatives. (Correlations between G and the parametric derivatives

can be transformed away by a change of variables.) We account for Coulomb interactions by using a canonical ensemble instead of a grand-canonical ensemble. Our results for the canonical ensemble are relevant for the analysis of recent experiments on chaotic quantum dots, where the conductance G is measured as a function of both the magnetic field and the shape of the quantum dot [8]. The grand-canonical results are relevant for experiments on microwave cavities [33,34]. Together with the theory provided here, such experiments can yield information on the distribution of delay times in chaotic scattering that can not be obtained by other means.

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